14.271: Industrial Organization I

Notes on Screening with a Continuum of Types

Michael Powell
Department of Economics, MIT
October 26th, 2006

1 Screening with a Continuum of Types

Consider a model of second-degree price discrimination in which there area continuum of types. That is, suppose consumers have preferences given by $V(q,\theta) - T(q)$ if they buy q units of a good and pay $T(q)$ dollars for it. Suppose θ is distributed with cdf $F(\theta)$ defined on $[\underline{\theta}, \bar{\theta}]$.

The monopolist wants to choose $T(q(.))$ and $q(\cdot)$ to

$$\max_{T(q(\cdot)),q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} [T(q(\theta)) - cq(\theta)] f(\theta) d\theta$$

subject to

$$V(q(\theta),\theta) - T(q(\theta)) \geq 0 \text{ for all } \theta \in [\underline{\theta}, \bar{\theta}] \tag{IR}$$

$$\theta = \underset{\tilde{\theta}}{\operatorname{argmax}} V\left(q\left(\tilde{\theta}\right),\theta\right) - T\left(q\left(\tilde{\theta}\right)\right). \tag{IC}$$

Define the value function

$$U(\theta) = \max_{\tilde{\theta}} V\left(q\left(\tilde{\theta}\right),\theta\right) - T\left(q\left(\tilde{\theta}\right)\right).$$

Then, by the envelope theorem, we have

$$\frac{\partial U(\theta)}{\partial \theta} = \frac{\partial V(q(\theta),\theta)}{\partial \theta}.$$

Thus, since

$$\begin{aligned} \int_{\underline{\theta}}^{\theta} \frac{\partial V(q(x),x)}{\partial \theta} dx &= \int_{\underline{\theta}}^{\theta} \frac{\partial U(x)}{\partial \theta} dx \\ &= U(\theta) - U(\underline{\theta}), \end{aligned}$$

by the fundamental theorem of calculus. We will necessariliy want that $U(\underline{\theta}) = V(q(\underline{\theta}),\underline{\theta}) - T(q(\underline{\theta})) = 0$, since otherwise, the monopolist could increase $T(q(\underline{\theta}))$ and increase its profits without affecting the IR of the lowest type. Thus,

$$U(\theta) = \int_{\underline{\theta}}^{\theta} \frac{\partial V(q(\theta),\theta)}{\partial \theta} d\theta,$$

and we have by definition

$$\begin{aligned} T(q(\theta)) &= V(q(\theta),\theta) - U(\theta) \\ &= V(q(\theta),\theta) - \int_{\underline{\theta}}^{\theta} \frac{\partial V(q(\theta),\theta)}{\partial \theta} d\theta. \end{aligned}$$

The monopolist's objective function then becomes

$$\begin{aligned} & \max_{q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left[V(q(\theta),\theta) - \int_{\underline{\theta}}^{\theta} \frac{\partial V(q(x),x)}{\partial \theta} dx - cq(\theta) \right] f(\theta) d\theta \\ = \; & \max_{q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} [V(q(\theta),\theta) - cq(\theta)] f(\theta) d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} \frac{\partial V(q(x),x)}{\partial \theta} dx f(\theta) d\theta. \end{aligned}$$

Using integration by parts, we can write the second term as

$$\int_{\underline{\theta}}^{\bar{\theta}} \left[\int_{\underline{\theta}}^{\theta} \frac{\partial V\left(q\left(x\right),x\right)}{\partial \theta} dx \right] \left[-f\left(\theta\right) d\theta\right].$$

Let $u = \int_{\underline{\theta}}^{\theta} \frac{\partial V(q(x),x)}{\partial \theta} dx$ and $dv = -f\left(\theta\right) d\theta$. Then

$$\begin{aligned} du &= \frac{\partial V\left(q\left(\theta\right),\theta\right)}{\partial \theta} d\theta \\ v &= 1 - F\left(\theta\right), \end{aligned}$$

so

$$\begin{aligned} \int_{\underline{\theta}}^{\bar{\theta}} \left[\int_{\underline{\theta}}^{\theta} \frac{\partial V\left(q\left(x\right),x\right)}{\partial \theta} dx \right] \left[-f\left(\theta\right) d\theta\right] &= \left[\int_{\underline{\theta}}^{\theta} \frac{\partial V\left(q\left(x\right),x\right)}{\partial \theta} dx \right] \left[1 - F\left(\theta\right)\right] \Bigg|_{\underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} \left[1 - F\left(\theta\right)\right] \frac{\partial V\left(q\left(\theta\right),\theta\right)}{\partial \theta} d\theta \\ &= -\int_{\underline{\theta}}^{\bar{\theta}} \frac{1 - F\left(\theta\right)}{f\left(\theta\right)} \frac{\partial V\left(q\left(\theta\right),\theta\right)}{\partial \theta} f\left(\theta\right) d\theta, \end{aligned}$$

since

$$\begin{aligned} \left[\int_{\underline{\theta}}^{\theta} \frac{\partial V\left(q\left(x\right),x\right)}{\partial \theta} dx \right] \left[1 - F\left(\theta\right)\right] \Bigg|_{\underline{\theta}}^{\bar{\theta}} &= \left[\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial V\left(q\left(x\right),x\right)}{\partial \theta} dx \right] \left[1 - F\left(\bar{\theta}\right)\right] \\ &\quad - \left[\int_{\underline{\theta}}^{\underline{\theta}} \frac{\partial V\left(q\left(x\right),x\right)}{\partial \theta} dx \right] \left[1 - F\left(\underline{\theta}\right)\right] \\ &= \left[\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial V\left(q\left(x\right),x\right)}{\partial \theta} dx \right] \cdot 0 - 0 \cdot 1 = 0. \end{aligned}$$

Thus, we can write our objective function as

$$\max_{q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left[V\left(q\left(\theta\right),\theta\right) - cq\left(\theta\right) - \frac{1 - F\left(\theta\right)}{f\left(\theta\right)} \frac{\partial V\left(q\left(\theta\right),\theta\right)}{\partial \theta} \right] f\left(\theta\right) d\theta.$$

We can then take our FOCs point by point

$$\left(q\left(\theta\right)\right) : \frac{\partial V\left(q\left(\theta\right),\theta\right)}{\partial q\left(\theta\right)} - c - \frac{1 - F\left(\theta\right)}{f\left(\theta\right)} \frac{\partial^2 V\left(q\left(\theta\right),\theta\right)}{\partial \theta} = 0,$$

or

$$\frac{\partial V\left(q\left(\theta\right),\theta\right)}{\partial q\left(\theta\right)} = c + \frac{1 - F\left(\theta\right)}{f\left(\theta\right)} \frac{\partial^2 V\left(q\left(\theta\right),\theta\right)}{\partial \theta}.$$

By assumption, $\frac{\partial^2 V(q(\theta),\theta)}{\partial \theta} > 0$, so we have that

$$\frac{\partial V\left(q\left(\theta\right),\theta\right)}{\partial q\left(\theta\right)} > c,$$

and thus $q\left(\theta\right) < q^{FB}\left(\theta\right)$ for each θ, except $\bar{\theta}$, where

$$\frac{\partial v\left(q\left(\bar{\theta}\right),\bar{\theta}\right)}{\partial q\left(\theta\right)} = c + \underbrace{\frac{1 - F\left(\bar{\theta}\right)}{f\left(\bar{\theta}\right)}}_{=0} \frac{\partial^2 V\left(q\left(\bar{\theta}\right),\bar{\theta}\right)}{\partial \theta} = c,$$

giving us $q\left(\bar{\theta}\right) = q^{FB}\left(\bar{\theta}\right)$, which is the standard "no distortion at the top" result of adverse selection models.