Revenue Equivalence Theorem:

Prop: Consider an N player independent private value (IPV) model $v_i \sim F[\underline{v}, \bar{v}]$. Suppose the object must be sold and all bidders must receive non-negative surplus ex post. (conditional on v_i) Then any auction satisfying (1) and (2) maximize's the seller's expected revenue.

$u = \begin{cases} v - p & \text{if win} \\ 0 & \text{if lose} \end{cases}$

- (1) The winning bidder in equilibrium has the highest value
- (2) The $\underline{v}$ type gets 0 surplus.

Corollary: First price, English, and second price yield the same revenue.

Exceptions to revenue equivalence:

1. Sellers usually do better with reserve price. e.g. $N=1$, $F \sim U[0,1]$. Optimum $p^* = \frac{1}{2}$
2. If bidders are risk averse: $\begin{cases} u(v_i - b_i) & \text{if win} \\ u(0) & \text{if lose} \end{cases}$

   $u' > 0$
   $u'' < 0$. Then a first price auction is better than an English auction.
   - English auction is equivalent to the risk neutral version.

- Suppose in the first price, all other bidders use $b^{*RN}(v)$. This would yield the same revenue. But:

$$\frac{d}{db} E[(v-b) I_{\{win\}}] \Big|_{b^{*RN}(v)} = 0$$

Raising bid by ε: gain $u[v - b^{*RN}(v)]\, p \cdot \varepsilon$
where $p = f_b(b^{*RN}(v))$

- $u(v-b^*) - u(v-b^*-\varepsilon)$ with probability $F \cdot \varepsilon$, where $F = F_b(b^*(v))^{N-1}$

- The gains are occurring on the steeper part of u. The losses are occurring on the flatter part of u.

- Want to exploit the risk aversion. By making them write their own price, they effectively will pay a risk premium. (Bidding higher is like buying insurance.)

3] With affiliated private values an English auction raises more revenue than a 1^{st} price auction. Affiliate private values:

$\frac{f(x_{-i}|x_i)}{f(x_{-i}'|x_i)}$ increasing in x_i when $x_{-i} > x_{-i}'$

Pf: • Consider strategies in terms of who a player will outbid. Choose z, the type you will outbid. $z = b^{*-1}(b)$ (ie bid $b^*(z)$)

• Focus on bidder utility $u^*(v) = u(z^*(v), v)$, where $u(z,v) = \underbrace{P(z,v)}_{\text{prob of winning if type } v \text{ and outbidding type } z}(v - \underbrace{E[z,v]}_{\text{payment to implement this strategy}})$

• The equilibrium will have $z^*(v) = v$

• Know that social surplus is the same in 1st and 2nd price auction.

$\Rightarrow$ Revenue highest when $E[u^*(v)]$ is lowest.

First price auction: $E[z,v]$ is independent of v.

$u^{*1st}(\underline{v}) = 0$.

$$\frac{du^*}{dv} = P(v,v) + (v - E[v,v])\frac{\partial P}{\partial v}$$

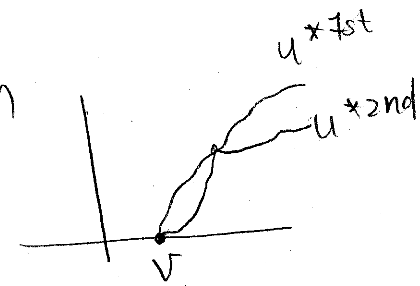
$$= P(v,v) + \frac{u^{*1st}(v)}{P(v,v)}\frac{\partial P}{\partial v}$$ since $u^{*1st}(v) = P(v,v)\cdot(v - E[v,v])$

Second price auction: $E[z, v]$ is increasing in v.

- conditional on my v being higher, the others' valuations are higher as well, and thus the expected payment to outbid type z is higher.
- $u^{*2nd}(\underline{v}) = 0$
- $$\frac{du^{*2nd}}{dv} = P(v,v) + \frac{u^{*2nd}(v)}{P(v,v)} \frac{\partial P}{\partial v} - P(v,v) \overbrace{\frac{\partial E}{\partial v}}^{+}$$

If $u^{*1st}(v) = u^{*2nd}(v)$, then

$$\frac{du^{*1st}}{dv} > \frac{du^{*2nd}}{dv}$$



$\Rightarrow u^{*2nd}(v) < u^{*1st}(v)$ almost everywhere

$\Rightarrow E[u^{*2nd}(v)] < E[u^{*1st}(v)]$.

Common Value auctions and Winner's Curse:

- anything being purchased for resale is a common value object.
- N bidders
- True value v to whoever wins. (e.g. $v \sim N(0,1)$)
- Bidders get private signals $s_i = v + \varepsilon_i$, where $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

- Write $w_i = E[v|s_i]$ (eg $w_i = \frac{1}{1+\sigma^2} s_i$)
- Whoever wins is going to be the person with the highest s_i.

Observations: • Don't bid w_i in 2nd price auction
- Bid less so you aren't disappointed.
- Bid $E[v|s_i, s_j < s_i \ \forall j \neq i, j', s_{j'} = s_i \text{ some } j']$

Multi-Unit auctions

- Lots of auctions have multiple units available
- 2 units to sell

	1 unit	2 units
consumer 1	60	$\boxed{100}$
consumer 2	30	40

efficient to sell both units to consumer 1

- Consider two handed English auction. If players don't use weakly dominated strategies and values are known, then the outcome is inefficient.
    - In equilibrium, each wins 1 unit.

Observations:

1. Bidder 2 must put one hand down at $p=10$.
2. Bidder 1 can then put down one hand at 10 and get $60 - 10 = 50$

3. To get both units, bidder 1 must keep hand up until $p = 30$

$\Rightarrow$ 1 gets $100 - 2(30) = 40$

4. 1 must drop out at 10.

Can use a Groves mechanism to get around this.

Complementarities: Cell phone company bidding for markets

	SC (Southern California!)	NC (Northern California!!)	Both
Bidder 1	v_1	0	v_1
Bidder 2	0	v_2	v_2
Bidder 3	10	10	30

$v_1 \sim U[10,20]$ Private information

$v_2 \sim U[10,20]$

Efficient to have 1 and 2 win if $v_1 + v_2 > 30$, 3 win if $v_1 + v_2 \leq 30$.

Thm: a sequential English auction is not efficient. Player 3 wins iff $v_1 \leq 15$. This is inefficient if $v_1 \leq 15$, $v_2 \geq 15$, $v_1 + v_2 \geq 30$ or if $v_1 \geq 15$, $v_2 \leq 15$, $v_1 + v_2 < 30$.

Pf: If 3 wins SC, 3's valuation for NC is 20
$\Rightarrow$ wins NC for sure.
$\Rightarrow$ 3's payoff is $30 - b - E[v_2]$
$= 15 - b$

If 3 drops out, payoff is zero. Thus, 3 stays in until $b = 15$ in SC.

Next time: Empirical auction papers.