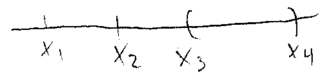
<u>Defn:</u> Let $x \in \mathbb{R}$, $\varepsilon > 0$, a <u>nbhd</u> of x is a set $N(x;\varepsilon) = \{y \in \mathbb{R} : |y - x| < \varepsilon\}$. ε is referred to as the <u>radius</u>.

<u>Defn:</u> Let $N(x;\varepsilon)$ be an ε-nbhd. Define $N^*(x;\varepsilon) = N(x;\varepsilon) \setminus \{x\}$ to be a <u>deleted nbhd</u>.

<u>Defn:</u> Let $S \subset \mathbb{R}$. a point $x \in \mathbb{R}$ is an <u>interior pt</u> of S if $\exists$ nbhd $N = N(x;\varepsilon) \subset S$. If $\forall$ nbhd N of x, $N \cap S \neq \emptyset$ and $N \cap (\mathbb{R} \setminus S) \neq \emptyset$, x is said to be a <u>boundary</u> point.

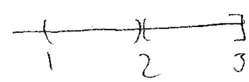
<u>Notation:</u> Let $S \subset \mathbb{R}$. Define $\text{int}\, S \equiv \{x : x \text{ is an int. pt. of } S\}$.
$\text{bd}\, S = \{x : x \text{ is a boundary point of } S\}$

Let $S = \{x_1\} \cup \{x_2\} \cup (x_3, x_4)$:



Clearly, $x_1, x_2 \notin \text{int}\, S$, $x_1, x_2 \in \text{bd}\, S$

Let $S = (1,2) \cup (2,3]$



$\text{int}\, S = (1,2) \cup (2,3)$
$\text{bd}\, S = \{1,2,3\}$

Let $S = \{1,2,3\}$
$\text{int}\, S = \emptyset$
$\text{bd}\, S = \{1,2,3\}$

<u>Defn:</u> $x \in \mathbb{R}$ is an <u>accomulation</u> pt if $\forall \varepsilon > 0$, $N^*(x;\varepsilon) \cap S \neq \emptyset$.

<u>Defn:</u> Let $S' = \{x : x \text{ is an accomulation pt. of } S\}$

<u>Defn:</u> If $x \in S \setminus S'$, x is an <u>isolated point</u>.

Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$
$S' = \{0\}$
The set of isolated points is S.

Let $S = (0,1)$
$S' = [0,1]$

Defn: The closure of S is defined by $\text{cl}\,S = S \cup S'$.

Defn: Let $S \subset \mathbb{R}$. If $\text{bd}\,S \subset S$, then S is said to be closed.
If $\text{bd}\,S \subset \mathbb{R} \setminus S$, then S is said to be open.

Thm: A set S is open iff $\mathbb{R} \setminus S$ is closed

Pf: ($\Leftarrow$) Let S^c be closed. Take $x \in S$. Then $x \notin S^c$. Then x is not an accumulation point of S^c. Thus $\exists$ a nbhd $N(x) \ni S^c \cap N$ is empty. That is, $N \subset S$.

($\Rightarrow$) Let S be open. Let $x \in \text{acc}\, S^c$. Then every $N(x)$ contains a point of S^c. Thus, $x \notin \text{int}\, S$ (since each nbhd intersects S^c). Thus, $x \in S^c$ (since S is open). That is, S^c contains all its accumulation points. $\Rightarrow S^c$ is closed.

Defn: A set S is compact if whenever $\exists\ \mathcal{F}$ a family of open sets s.t. $S \subset \bigcup_{F \in \mathcal{F}} F$, $\exists$ a finite subset $\mathcal{F}^*$ s.t. $S \subset \bigcup_{F \in \mathcal{F}^*} F$.

Let $S = (0,1)$
$\mathcal{F} = \{A_n = (\frac{1}{n}, 2) : n \in \mathbb{N}\}$

Clearly, $\bigcup_{n=1}^{\infty} A_n \supset S$, but $\nexists$ a finite number of A_n's containing S.

Let $\{A_{n_1}, \dots, A_{n_m}\}$ be an arbitrary finite subcover.
Then $\frac{1}{n_m + 1} \notin \bigcup_{k=1}^{m} A_{n_k}$.

Let $S=\{x_1,\dots,x_n\}$. Let $\mathcal{F}=\{A_\alpha:\alpha\in\mathcal{A}\}$ be any open cover. For each $i=1,\dots,n$, $\exists$ a set $A_i\in\mathcal{F}$ s.t. $x_i\in A_i$. Take $\mathcal{F}^*=\bigcup_{i=1}^{n} A_i$. Then $\mathcal{F}^*\supset S$. Thus, S is compact.

Thm: (Heine-Borel) Let $S\subset\mathbb{R}$. S is compact iff S is closed and bounded.

Metric spaces

Defn: Let $S\neq\emptyset$. Then let $d: S\times S\to\mathbb{R}$ satisfy $\forall x,y,z\in S$

i) $d(x,y)\geq 0$

ii) $d(x,y)=0$ iff $x=y$

iii) $d(x,y)=d(y,x)$

iv) $d(x,y)\leq d(x,z)+d(z,y)$

Then we say that d is a metric and (S,d) is a metric space.

eg. Let $S=\mathbb{R}^2$, $x=(x_1,x_2)\in S$, $y=(y_1,y_2)\in S$

i) $d_1(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

ii) $d_2(x,y)=\max\{|x_1-y_1|,|x_2-y_2|\}$

iii) $d_3(x,y)=|x_1-y_1|+|x_2-y_2|$

Defn: Let (S,d) be a metric space. Let $x\in S$, $\varepsilon>0$. Then the ε-nbhd of x is $N(x;\varepsilon)=\{y\in S: d(x,y)<\varepsilon\}$

Thm: Let (S,d) be a metric space. Then any nbhd of $x\in S$ is an open set.

Pf: Let $y \in N(x;\varepsilon)$. Take $\delta = \varepsilon - d(x,y)$. Then, let $z \in N(y;\delta)$. $d(x,z) \leq d(x,y) + d(y,z)$
$$< d(x,y) + \varepsilon - d(x,y) = \varepsilon$$

Thus, $N(y;\delta) \subseteq N(x;\varepsilon) \Rightarrow N(x;\varepsilon)$ is open. □

Defn: $T \subset S$ is bdd if $T \subset N(x;M)$ for some $x \in S, M > 0$

Thm: Let $T \subset S$ be compact. Then
- T is closed and bounded
- Every infinite subset of T contains an accumulation point. (Bolzano-Weierstrass or sequential compactness.)

Sequences

Defn: a sequence is a function whose domain is the natural numbers. $s: \mathbb{N} \to B$. We can enumerate the elements: $(s_1, \dots, s_n, \dots) = (s_n)_{n\in\mathbb{N}}$

Defn: Let $B = \mathbb{R}$. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. We say $(s_n)_{n\in\mathbb{N}}$ converges to $s \in \mathbb{R}$ if $\forall \varepsilon > 0 \; \exists N(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \geq N$, $|s_n - s| < \varepsilon$. We refer to s as the limit of $(s_n)_{n\in\mathbb{N}}$. We write

$\lim s_n = s$.

Thm: Every convergent sequence is bounded

Pf: Let (s_n) be s.t. $\lim s_n = s$. Then $\exists \varepsilon > 0$ s.t. $\forall n \geq N(\varepsilon)$, $|s_n - s| < \varepsilon \Rightarrow |s_n| \leq \varepsilon + |s| \equiv s^*$.
Let $M = \max\{s_1, \dots, s_{N(\varepsilon)}, s^*\}$. Then $\forall n, |s_n| \leq M$.
Thus, (s_n) is bounded □.

<u>Thm</u>: Let $(s_n), (t_n)$ be s.t. $\lim s_n = s$, $\lim t_n = t$. Then $\lim(s_n + t_n) = s + t$

<u>Pf</u>: $s_n \to s \Rightarrow \forall \varepsilon > 0\ \exists N_s(\varepsilon)$ s.t. $\forall n \geq N_s(\varepsilon)$, $|s_n - s| < \varepsilon/2$

$t_n \to t \Rightarrow \forall \varepsilon > 0\ \exists N_t(\varepsilon)$ s.t. $\forall n \geq N_t(\varepsilon)$, $|t_n - t| < \varepsilon/2$

Let $\varepsilon > 0$ be arbitrary. Choose $N(\varepsilon) = \max\{N_s(\varepsilon), N_t(\varepsilon)\}$

Then $\forall n \geq N(\varepsilon)$, $|s_n + t_n - s + t| \leq |s_n - s| + |t_n - t|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$

<u>Thm</u>: Let $(s_n), (t_n)$ be s.t. $\lim s_n = s$, $\lim t_n = t$. Then

i) $\forall k \in \mathbb{R}$, $\lim k s_n = k \lim s_n = ks$

ii) $\lim s_n t_n = (\lim s_n)(\lim t_n) = s \cdot t$

iii) $\lim \frac{s_n}{t_n} = \frac{\lim s_n}{\lim t_n} = \frac{s}{t}$ if $t_n \neq 0\ \forall n$, $t \neq 0$

<u>Thm</u>: Let (s_n) be s.t. $\lim s_n = s$, $\lim s_n = s'$, then $s = s'$.

<u>Pf</u>: Let $s > s'$. Then let $\varepsilon = \frac{s - s'}{2}$. Then $\exists N(\varepsilon)$ s.t. $\forall n \geq N(\varepsilon)$, $|s_n - s| < \varepsilon$ and $|s_n - s'| < \varepsilon$.

$\Rightarrow \varepsilon - s < s_n < \varepsilon + s$ and $\varepsilon - s' < s_n < \varepsilon + s'$

$\Rightarrow s_n > s_n$, which is a contradiction. Thus, $s = s'$

<u>Thm</u>: (s_n) <u>diverges</u> to $+\infty$ $(-\infty)$ if $\forall M > 0\ \exists N(M)$ s.t. $\forall n \geq N(M)$, $s_n \geq M$. $(s_n \leq -M)$.

<u>Defn</u>: (s_n) is <u>increasing</u> if $n' > n \Rightarrow s_{n'} > s_n$
(s_n) is <u>decreasing</u> if $n' > n \Rightarrow s_{n'} < s_n$
(s_n) is <u>monotone</u> if it is increasing or decreasing.

<u>Thm</u>: If (s_n) is monotone, it is convergent iff it is bounded.

<u>Defn</u>: (s_n) is a <u>Cauchy sequence</u> if $\forall \varepsilon > 0, \exists N(\varepsilon)$ s.t. $\forall n,m \geq N$, $|s_n - s_m| < \varepsilon$.

<u>Lemma</u>: Every convergent sequence is a Cauchy sequence.

<u>Pf</u>: Let (s_n) be s.t. $\lim s_n = s$. Then $\forall \varepsilon > 0\ \exists N(\varepsilon)$. s.t. $\forall n \geq N$, $|s_n - s| < \frac{\varepsilon}{2}$. Take $n, m \geq N$. Then
$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$

<u>Lemma</u>: Every Cauchy sequence is bounded.

<u>Thm</u>: (s_n) is Cauchy iff (s_n) is convergent.

<u>Defn</u>: a sequence (s_n) is <u>contractive</u> if $\exists k$ s.t. $0 < k < 1$ s.t. $|s_{n+2} - s_{n+1}| \leq k|s_{n+1} - s_n|\ \forall n \in \mathbb{N}$

<u>Thm</u>: Every contractive sequence is convergent.

<u>Pf</u>: $|s_m - s_n| \leq |s_m - s_{m-1}| + \ldots + |s_{n+1} - s_n|$
$\leq (k^{m-n-1} + k^{m-n-2} + \ldots + k + 1)|s_{n+1} - s_n|$
$= k^{n-1}(k^{m-n-1} + k^{m-n-2} + \ldots + k + 1)|s_2 - s_1|$
$\leq \dfrac{k^{n-1}}{1-k}|s_2 - s_1|$ but $k^{n-1} \to 0$ since $0 < k < 1$

Thus $|s_m - s_n| \to 0$ by squeeze thm.

Therefore $\forall \varepsilon > 0 \; \exists N(\varepsilon)$ s.t. $\forall m > n \geq N$, $|s_m - s_n| < \varepsilon$
i.e. (s_n) is Cauchy. Thus, (s_n) is convergent.

Defn: Let $f: D \to \mathbb{R}$. Let $c \in \operatorname{acc} D$. We say $L = \lim_{x \to c} f(x)$ if $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$

Thm: Let $c \in \operatorname{acc} D$. Then $\lim_{x \to c} f(x) = L$ iff $\forall$ nbhd V of L, $\exists$ some deleted nbhd U of c s.t. $f(U \cap D) \subset V$.

Thm: Let $c \in \operatorname{acc} D$. $\lim_{x \to c} f(x) = L$ iff $\forall (s_n)$ s.t. $s_n \to c$, $\lim f(s_n) = L$.