Relevant sections of Folland: Chapter 1: 1.1–1.5, Chapter 2: 2.1–2.6 (not including the change of variables formula), Chapter 3: 3.1–3.4
↳ Closed book test

Review:

Example 1: Let $f \in L^1_{loc}(\mathbb{R})$ be s.t. $f(x+\frac{1}{n}) \geq f(x)$ for all $n \in \mathbb{N}$ and a.a. x. Show that $(*)\ f(x+a) \geq f(x)\ \forall a \in \mathbb{R}^+$ and a.a. x.

Solution: If $q = \frac{m}{n} \in \mathbb{Q}^+$, then $f(x+q) = f(x+\frac{m}{n}) \geq f(x+\frac{m-1}{n}) \geq \dots \geq f(x)$

Thus $(*)$ holds when $a \in \mathbb{Q}^+$. If f is continuous, then $(*)$ follows $\forall a \in \mathbb{R}^+$

In the general case, consider $\underbrace{f_s(x)}_{\text{continuous fcn of } x} = \int_x^{x+s} f(y)\,dy \quad , s>0$

If $f \in L^1_{loc}$, then $x \mapsto \int_a^x f(t)\,dt$ is continuous by dominated convergence.

Then, $\underset{q \in \mathbb{Q}^+}{f_s(x+q)} = \int_{x+q}^{x+q+s} f(y)\,dy = \int_x^{x+s} \underbrace{f(y+q)}_{\geq f(y) \text{ for a.a. } y}\,dy \geq \int_x^{x+s} f(y)\,dy = f_s(x)$.

We get $f_s(x+q) \geq f_s(x) \quad \forall x \in \mathbb{R} \quad \forall q \in \mathbb{Q}$

If $a \in \mathbb{R}^+$, take a sequence $q_n \in \mathbb{Q}^+$ with $q_n \to a$. Then $f_s(x+q_n) \geq f_s(x)$. Taking limits, by continuity of f_s, $f_s(x+a) \geq f_s(x)$

We get $\underbrace{\int_{x+a}^{x+a+s} f(y)\,dy}_{= \int_x^{x+s} f(y+a)\,dy} - \int_x^{x+s} f(y)\,dy \geq 0$

$\Rightarrow \int_x^{x+s} [f(y+a) - f(y)]\,dy \geq 0 \quad \forall x, \ \forall s > 0$.

By Lebesgue differentiation theorem, $f(x+a) - f(x) = \lim_{s \to 0} \frac{1}{s} \int_x^{x+s} [f(y+a) - f(y)]\,dy \geq 0$ for a.a. x. □

Another application of the Lebesgue differentiation theorem.
If $f \in L^1(\mathbb{R}^n)$ and $\int_E f(x)\,dx = 0\ \forall$ measurable $E \Rightarrow f(x) = 0$ a.e.

Example 2: Let $f \in L^1(\mathbb{R}^n)$. Show that $\lim_{|h| \to \infty} \int |f(x+h) - f(x)|\,dx = 2 \int |f(x)|\,dx$

Solution: Let us assume first that f vanishes outside a bdd set.

If $M_1 = \operatorname{supp} f$, then M_1 is compact.

Then the support of $x \mapsto f(x+h)$ is $M_1 - h = M_2$. Now,

$$\int |f(x+h) - f(x)|\,dx = \int_{\underbrace{M_1 \cup M_2}_{\text{disjoint for } h \text{ is large enough}}} |f(x+h) - f(x)|\,dx = \int_{M_1} |f(x+h)|\,dx + \int_{M_2} |f(x)|\,dx$$

$$= \int_{M_1} |f| + \int_{M_2} |f| = 2 \int |f|\,dx$$

In the general case, take $f_j = \chi_{\{|x| \le j\}} f \to f \in L^1$ by dominated convergence. We get $\|f(\cdot + h) - f\|_{L^1} - 2\|f\|_{L^1}$

$$= \|f(\cdot+h) - f_j(\cdot+h) + f_j(\cdot+h) - f_j + f_j - f\|_{L^1} - 2\|f - f_j + f_j\|_{L^1}$$

$$\le \underbrace{\|f(\cdot+h) - f_j(\cdot+h)\|_{L^1}}_{\to 0 \text{ dom conv}} + \underbrace{\|f_j(\cdot+h) - f_j\|_{L^1}}_{\to 2\|f_j\|_{L^1}} + \underbrace{\|f_j - f\|_{L^1}}_{\to 0 \text{ dom conv.}} + \underbrace{2\|f - f_j\|_{L^1}}_{\to 0 \text{ dom conv}} - 2\|f_j\|_{L^1}$$

$\to 0$

Thus $\|f(\cdot+h) - f\|_{L^1} \to 2\|f\|_{L^1}$ as $|h| \to \infty$ □

Example 3: Show that if $f \in L^1(\mathbb{R})$, then $\sum_{n=1}^{\infty} n \int_n^{n+\frac{1}{n}} |f(x+y)|\,dy < +\infty$ for a.a. x

Soln: Need to show $\sum_{n=1}^{\infty} n \int_a^{a+1} \left(\int_n^{n+\frac{1}{n}} |f(x+y)|\,dy \right) dx$