Convergence:

① convergence in probability: $\forall \varepsilon > 0$

$$\lim_{n\to\infty} Pr(|X_n - X| \geq \varepsilon) = 0$$

Equivalently, write $\operatorname{plim}_{n\to\infty} X_n = X$ or $X_n \xrightarrow{p} X$

② almost sure convergence (converges almost surely)
(convergence almost everywhere)

$X_n \xrightarrow{a.s.} X$

$$P(\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)) = 1$$

The set of elements for which $X_n(\omega) \xrightarrow[n\to\infty]{} X(\omega)$ pointwise is of measure 1.

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X$$
$$\not\Leftarrow$$

Example

$$X_n = \begin{cases} 0 & \text{with prob } \frac{1}{n} \\ 1 & \text{with prob } 1 - \frac{1}{n} \end{cases}$$

$E[X_n] = 1 - \frac{1}{n}$ $\qquad Var[X_n] = \frac{1-3n}{n^2} \xrightarrow{n\to\infty} 0$

$\lim_{n\to\infty} E[X_n] = 1$

If $E[X_n] \xrightarrow{n\to\infty} E[X]$ and $Var(X_n) \xrightarrow{n\to\infty} 0$, then $X_n \xrightarrow{p} X$.

Thus, $X_n = \begin{cases} 0 & \text{w/ prob } \frac{1}{n} \\ 1 & \text{w/ prob } 1-\frac{1}{n} \end{cases} \xrightarrow{p} 1 = X$

It can be shown that $X_n \overset{a.s.}{\not\to} X$

Example. Suppose $Y_n \sim B(n, p)$. Show that ① $\operatorname{plim} \frac{Y_n}{n} = p$

② $1 - Y_n/n \xrightarrow{p} 1-p$

③ $\frac{Y_n}{n}\left(1 - \frac{Y_n}{n}\right) \xrightarrow{p} p(1-p)$

(1) $\Pr[|\bar{X}-\mu| \geq \varepsilon] \leq \frac{\sigma^2}{\varepsilon^2}$ by Chebyshev

$E[Y_n] = np$

$Var[Y_n] = np(1-p)$

$E[Y_n^2] = Var(Y_n) + [E[Y_n]]^2 = np(1-p) + n^2p^2$

$\frac{Y_n}{n} \xrightarrow{P} p \iff \lim_{n\to\infty} \Pr[|Y_n/n - p| \geq \varepsilon] = 0$

$\iff \lim_{n\to\infty} \Pr[|Y_n - np| \geq n\varepsilon] = 0$

$\iff \lim_{n\to\infty} \Pr[|Y_n - np| \geq n\varepsilon] \leq \lim_{n\to\infty} \frac{np(1-p)}{n^2\varepsilon^2} = \lim_{n\to\infty} \frac{p(1-p)}{n\varepsilon^2} = 0$

Thus, $\operatorname{plim}_{n\to\infty} \frac{Y_n}{n} = p$

(3) $\Pr\left[\left|\frac{Y_n}{n}\left(1-\frac{Y_n}{n}\right) - p(1-p)\right| \geq \varepsilon\right]$

$= \Pr\left[\left|\frac{Y_n}{n} - p - \left(\frac{Y_n^2}{n^2} - p^2\right)\right| \geq \varepsilon\right]$

$\left|\frac{Y_n}{n} - p\right| + \left|\frac{Y_n^2}{n^2} - p^2\right| \geq \left|\frac{Y_n}{n} - p - \left(\frac{Y_n^2}{n^2} - p^2\right)\right| \geq \varepsilon$

$\iff \left|\frac{Y_n}{n} - p\right| \geq \frac{\varepsilon}{2}$ or $\left|\frac{Y_n^2}{n^2} - p^2\right| \geq \frac{\varepsilon}{2}$

Thus, $\Pr\left[\left|\frac{Y_n}{n} - p - \left(\frac{Y_n^2}{n^2} - p^2\right)\right| \geq \varepsilon\right]$

$\leq \underbrace{\Pr\left[\left|\frac{Y_n}{n} - p\right| \geq \frac{\varepsilon}{2}\right]}_{\to 0 \text{ from part a}} + \Pr\left[\left|\frac{Y_n^2}{n^2} - p^2\right| \geq \frac{\varepsilon}{2}\right]$

Note $\frac{Y_n^2}{n^2} - p^2 = \left(\frac{Y_n}{n} - p\right)^2 + 2p\left(\frac{Y_n}{n} - p\right)$. Thus,

$\Pr\left[\left|\frac{Y_n^2}{n^2} - p^2\right| \geq \frac{\varepsilon}{2}\right] \leq \underbrace{\Pr\left[\left|\frac{Y_n}{n} - p\right|^2 \geq \frac{\varepsilon}{4}\right]}_{\to 0 \text{ from part a}} + \underbrace{\Pr\left[\left|2p\left(\frac{Y_n}{n} - p\right)\right| \geq \frac{\varepsilon}{4}\right]}_{\to 0 \text{ from part a.}}$

Some theorems:

① $X_n \xrightarrow{p} X,\ Y_n \xrightarrow{p} Y \Rightarrow X_n + Y_n \xrightarrow{p} X + Y$

② $X_n \xrightarrow{p} X$ and $a \in \mathbb{R} \Rightarrow aX_n \xrightarrow{p} aX$

③ $X_n \xrightarrow{p} a$ and $g(\cdot)$ continuous at $a \Rightarrow g(X_n) \xrightarrow{p} g(a)$

③′ $X_n \xrightarrow{p} X$, and $g(\cdot)$ is continuous on $\text{ran}(X) \Rightarrow g(X_n) \xrightarrow{p} g(X)$

④ $X_n \xrightarrow{p} X,\ Y_n \xrightarrow{p} Y \Rightarrow X_n Y_n \xrightarrow{p} XY$

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o_p vs O_p

$\downarrow$ little "oh pee" $\quad\downarrow$ big "oh pee"

o_p : convergence in probability to zero "little oh"

$X_n = o_p(n^{-1/2}) \Rightarrow X_n \sqrt{n} \xrightarrow{p} 0$

$X_n = o_p(c) \Rightarrow \frac{X_n}{c} \xrightarrow{p} 0$

O_p : bounded in probability

$\lim_{n \to \infty} \Pr[\,|X_n| \le B\,] = 1$ for some $B > 0$.

$X_n \xrightarrow{p} X \iff X_n = X + o_p(1)$

$o_p + o_p = o_p$

$o_p + O_p = O_p$

$o_p \cdot o_p = o_p$

$o_p \cdot O_p = o_p$

Laws of large numbers.

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \underset{a.s.}{\xrightarrow{p}} \mu = E[X_i]$$

WLLN

SLLN

Conditions:

WLLN $\begin{cases} (1) \; X_1, \ldots, X_n \text{ iid} \\ (2) \; E[X_i] = \mu \text{ and } Var(X_i) = \sigma^2 < +\infty \end{cases}$

SLLN $\{ (2)' \; E[|X_i|] < +\infty$

$$E[|X_i|] < +\infty \Rightarrow E[X_i] < +\infty \text{ and } Var(X_i) < +\infty$$

applications:

(1) Compute probability

example: $X_1, \ldots, X_n \overset{iid}{\sim} N(0,1)$

$$Y = \sum_{i=1}^{n} X_i$$

$$Z = X_1 + X_2^2 + X_3^{1/2} + \log X_4 + e^{X_5} + \cdots + \frac{1}{X_n}$$

$Pr(Y \leq 0.8) = ?$ $Pr(Z \leq 1) = ?$

How do we calculate these?

$$Pr(Y \leq y) = E[I(Y \leq y)] \overset{p}{\approx} \frac{1}{n} \sum_{i=1}^{n} I(Y \leq y)$$

Use Matlab

```
>> clear all
>> Y = zeros (5000, 1)
>> Z = zeros (5000, 1)
>> for  i = 1:5000
>>      x = randn (1000, 1)
>>      y = sum (x)
>>      z = x(1) + x(2)^2 + ...
>>      Y(i) = y
>>      Z(i) = z
>> end
>> P1 = mean((Y<=0.8))
>> P2 = mean((Z<=1))
```

indication parenthesis

(2) Compute expectation

$E[Y]$, $E[Y^3]$, $E[Z^{1/2}]$

$$E[Y] \approx \frac{1}{n}\sum_{i=1}^{n} Y_i = a$$

$$E[Y^3] \approx \frac{1}{n}\sum_{i=1}^{n} Y_i^3 = b$$

$$E[Z^{1/2}] \approx \frac{1}{n}\sum_{i=1}^{n} Z_i^{1/2} = c$$

```
>> a = mean(Y)
>> b = mean(Y.^3)
>> c = mean(Z.^(1/2))
```

(3) Generate random variables

i) cdf F is known and strictly increasing

$\Rightarrow Y = F^{-1}(u)$ where $u \sim U(0,1)$

to generate $u_1, \ldots, u_n$,

```
>> rand(n,1)
```

ii) unknown or hard to compute F^{-1}

e.g. $F(x) = e^{e^x} + \sqrt{x+3}$

$x = F^{-1}(\cdot)$?

For this, we must use importance sampling

$$F(t) = E\left[I(X \le t) \frac{f(x)}{g(x)} \right]$$

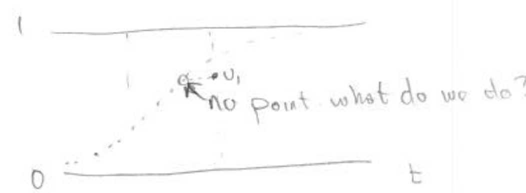
$$\overset{p}{\approx} \frac{1}{n}\sum_{i=1}^{n} I(X_i \le t) \frac{f(x_i)}{g(x_i)}$$

easy and known

Given some $t_1 \Rightarrow \widehat{F(t_1)}$

$\vdots$

$t_n \Rightarrow \widehat{F(t_n)}$



This will trace out the cdf F.

Then take closest $t \ni |\widehat{F^{-1}(t)} - u_i|$ is minimized.

$$\frac{\widehat{F(t_A)} - u_1}{u_1 - \widehat{F(t_B)}} = \frac{t_A - t_1}{t_1 - t_B} \Rightarrow t_1$$

Example: Computer can generate standard normal $N(0,1)$.

How to generate $f(x) = \sqrt{x+1}\, e^{-2x}$ $x > 0$?

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

$$F(t) = \int_{-\infty}^{\infty} I(0 \le X \le t) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} I(0 \le X \le t) \frac{f(x)}{\varphi(x)} \varphi(x)\,dx \qquad \text{where } \varphi(x) \text{ is pdf of } N(0,1)$$

$$= E\left[I(0 \le X \le t) \frac{f(X)}{\varphi(X)} \right]$$

Laws of large numbers

$$\overset{p}{\approx} \frac{1}{n} \sum_{i=1}^{n} I(0 \le x_i \le t) \frac{f(x_i)}{\varphi(x_i)} \qquad \leftarrow \text{simulate now}$$

$t = -500 : 0.01 : 500 \Rightarrow \widehat{F(t)}$