$\underset{n\times 1}{X} \sim N(0, I_n)$

$\Rightarrow X'X \sim \chi^2(n)$

Proof: $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ $x'x = \sum_{i=1}^{n} x_i^2$

$\underset{n\times n}{A} = (a_{ij})$

$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$

$\text{trace}(A+B) = \text{trace}(A) + \text{trace}(B)$

$\text{trace}(\underset{m\times n}{A}\underset{n\times m}{B}) = \text{trace}(\underset{n\times m}{B}\underset{m\times n}{A})$ (*) Prove this on your own.

Lemma: Suppose P is idempotent (ie. $P = P\cdot P$) and symmetric (ie $P = P'$). Then all eigenvalues of P are either 0 or 1.

Proof: If λ is an eigenvalue, then $\exists x$ s.t. $Px = \lambda x$.

Therefore $\lambda x = Px = (PP)x = P(Px) = P\lambda x = \lambda(Px) = \lambda^2 x$

and thus $\lambda = \lambda^2 \Rightarrow \lambda \in \{0,1\}$

Theorem: Suppose $\underset{n\times n}{P}$ is symmetric and idempotent. Suppose $X \sim N(\underset{n\times 1}{0}, I_n)$. Then, $\underset{1\times n}{X'}\underset{n\times n}{P}\underset{n\times 1}{X} \sim \chi^2(\text{trace}(P))$

Proof: Define $\Lambda = \begin{array}{c} \begin{matrix} m & n-m \end{matrix} \\ \begin{bmatrix} 1 & & & 0 \\ & 1 & 0 & \\ 0 & & \ddots & \\ & & & 0 \end{bmatrix} \end{array}$. Then $P = C'\Lambda C$ with $C'C = CC' = I_n$

Define $Y \equiv CX \sim N(0, CC') = N(0, I_n)$.

$X'PX = X'C'\Lambda CX = (CX)'\Lambda(CX) = Y'\Lambda Y$

Assume that $\text{trace}(\Lambda) = m$. Then,

$Y'\Lambda Y = [Y_1 \; Y_n]\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix}\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \sum_{i=1}^{m} Y_i^2 \sim \chi^2(m)$.

Finally, $\text{trace}(P) = \text{trace}(\underbrace{C'}_{A}\underbrace{\Lambda C}_{B}) = \text{trace}(\Lambda CC') = \text{trace}(\Lambda) = m$. $\square$

The remainder of this course will be devoted to inverse probability. That is, statistics. (Prob - game of gods, Stats - game of human beings)

Suppose X is a random vector $X \overset{pdf}{\sim} \{f_X(x;\theta_0);\ \theta_0 \in \Theta\}$ ≡ family of pdfs.
Let $u(X)$ be a statistic. Note u cannot be a fcn of θ.
You want to find out θ_0.

Defn: An estimator is said to be unbiased if

$$E[u(X)] = \int u(x) f(x;\theta)\,dx = \theta$$

← arbitrary value of θ.

Notation: We will denote the "truth" with a zero subscript.
You are "correct" if $E[u(X)] = \int u(x) f(x;\theta_0)\,dx = \theta_0$

There is a semantic difference b/t being "correct" and being "unbiased."

Consistent estimator.
If $\underset{n\to\infty}{\text{plim}}\ u(X) = \theta$, then $u(X)$ is called a consistent estimator.

[Spse $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$. $u(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then if $X' = \begin{pmatrix} X_1 \\ \vdots \\ X_{n+1} \end{pmatrix}$, we cannot use the "same" estimator. The correct notation should be $u_n(X)$.]

Maximum likelihood estimator (MLE)

Likelihood

Spse $X_1, \dots, X_n \overset{iid}{\sim} f(x;\theta)$

(joint) pdf $\Rightarrow \prod_{i=1}^{n} f(x_i;\theta)$ ← function in x_i

(joint) likelihood: $\prod_{i=1}^{n} f(X_i;\theta)$ ← function in θ.

Suppose $X_1, X_2 \overset{iid}{\sim} N(\theta, 1)$

$$f(x_1,x_2;\theta) = \frac{1}{2\pi}\exp\left[-\frac{(x_1-\theta)^2+(x_2-\theta)^2}{2}\right] = g(x_1,x_2)$$

Fix x_1^*, x_2^*

$$f(x_1^*,x_2^*;\theta) = \frac{1}{2\pi}\exp\left[-\frac{(x_1^*-\theta)^2+(x_2-\theta)^2}{2}\right] = h(\theta)$$

MLE:

$\underline{X} \overset{pdf}{\sim} f(x;\theta)$

$L(\theta) = f(\underline{X};\theta)$

$\underset{\theta}{\text{argmax}}\ L(\theta) = u(\underline{X}) \leftarrow$ MLE

Proposition: Under "regularity conditions," MLE is consistent, although it may not be unbiased.

Example: Let $X_i \overset{iid}{\sim} N(\theta, 1)$ $i \in \{1, \dots, n\}$

$$L(\theta) = \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{\sum_{i=1}^{n}(X_i - \theta)^2}{2}\right]$$

$$\hat{\theta} = \underset{\theta}{\text{argmax}}\ L(\theta) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

(*) Know the derivation.

(Invariance principle)

$h(\theta)$ - you are interested in this. Spse $h(\theta)$ is 1-1.

$\widehat{h(\theta)} = h(\hat{\theta})$

Suppose $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 1) = N(\varphi^3, 1)$ if $\theta = \varphi^3$

$\hat{\varphi} = \hat{\theta}^{1/3}$. We can find $\hat{\varphi}$ by calculating $\hat{\theta}$. This is all the theorem says.

Method of Moments estimator

$X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 1)$

$\theta = E[X_i] \approx \frac{1}{n}\sum_{i=1}^{n} X_i = \hat{\theta}$

Suppose $X_i \sim \text{Gamma}(\alpha, \beta)$.

$E[X_i] = \alpha\beta$

$Var(X_i) = \alpha\beta^2$

$\Rightarrow E[X_i^2] = \alpha^2\beta^2 + \alpha\beta^2$

Suppose we want to find α and β.

Solve for $\hat{\alpha}, \hat{\beta}$ solving:

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \hat{\alpha}\hat{\beta}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \hat{\alpha}^2\hat{\beta}^2 + \hat{\alpha}\hat{\beta}^2$$