$$\text{median} = \underset{b}{\operatorname{argmin}}\ E[|X-b|]$$

$$\begin{aligned}
E[|X-b|] &= E[(X-b)\cdot 1(X>b) + (b-X)1(X\le b)] \\
&= E[(X-b)1[X>b]] + E[(b-X)1(X\le b)] \\
&= E[X 1(X>b) - b\cdot 1(X>b)] + E[b 1(X\le b) - X 1(X\le b)] \\
&= E[\underbrace{\quad\cdots\quad}_{\text{not fcn of } b}] - b[E[1(X>b)] - E[1(X\le b)]]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial E[|X-b|]}{\partial b} &= E[1(X\le b)] - E[1(X>b)] = 0 \\
&\Rightarrow Pr(X\le b) - Pr(X>b) = 0 \\
&\Rightarrow Pr(X\le b) = Pr(X>b) \\
&\Rightarrow b=m.
\end{aligned}$$

Last time: $E[|X-b|] = E[|X-m|] + 2\int_m^b (b-m)f(x)dx$

Example: X is positive r.v. Show

① $E\left[\frac{1}{X}\right] \ge \frac{1}{E[X]}$

② $E[X^3] \ge (E[X])^3$

Jensen's inequality:

① $u(x) = \frac{1}{x}$ $(x>0)$ convex since $u''(x) = 2x^{-3} > 0$

Thus, by Jensen's inequality,

$$E[u(X)] \ge u(E[X])$$

$$\Rightarrow E\left[\frac{1}{X}\right] \ge \frac{1}{E[X]}$$

(2) Let $u(x) = x^3$ $\qquad u''(x) = 6x > 0 \quad \forall x > 0$

Thus, u is convex

$$E[u(X)] \geq u(E[X])$$

$$E[X^3] \geq (E[X])^3$$

Independence:

$$f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$$

$$E[XY] = E[X]E[Y] \Leftarrow \text{independence}$$
$$\not\Rightarrow \text{independence}$$

$$Cov(X, Y) = 0 \Leftarrow \text{independence}$$
$$\not\Rightarrow \text{independence}$$

Example:

$$f(x,y) = \begin{cases} 3x & 0 < y < x < 1 \\ 0 & \text{else} \end{cases} \qquad X \perp Y?$$

$$f_X(x) = \int_0^x 3x\,dy = 3x \int_0^x dy = 3x^2$$

$$f_Y(y) = \int_y^1 3x\,dx = \left(\frac{3x^2}{2}\right)\Big|_y^1 = \left(\frac{3}{2} - \frac{3y^2}{2}\right)$$

$$f_X(x) f_Y(y) = 3x^2\left(\frac{3}{2} - \frac{3y^2}{2}\right) \neq 3x = f(x,y) \Rightarrow X \not\perp Y$$

Example: X, Y rvs

$$f(x,y) = \begin{cases} 8xy & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

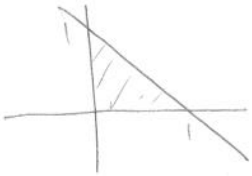
$Z = \frac{X}{Y}$. Find pdf of Z

$$G(z) = Pr(Z \leq z) = Pr\left(\frac{X}{Y} \leq z\right) = Pr(X \leq zY)$$

$$= \int_0^1 \int_0^{yz} f(x,y)\,dx\,dy$$

$0 \le x_1 \le x_2 \le \dots \le x_n \le 1$

$$Q(x_1, \dots, x_n) = \int_0^1 \int_0^{x_n} \int_0^{x_{n-1}} \cdots \int_0^{x_3} \int_0^{x_2} f(\dots)\, dx_1\, dx_2 \dots dx_{n-1}\, dx_n$$



$x + y < 1$
$x > 0,\ y > 0$ $\Rightarrow Pr(X+Y \le 1) = \int_0^1 \int_0^{1-y} f(x,y)\, dx\, dy$

$0 < x < 1-y < 1$

$$G(\quad z) = \int_0^1 \int_0^{yz} f(x,y)\, dx\, dy$$
$$= \int_0^1 \int_0^{yz} 8xy\, dx\, dy$$
$$= \int_0^1 \left[4x^2 y \Big|_0^{yz} \right] dy$$
$$= \int_0^1 4y^3 z^2\, dy = z^2 \int_0^1 4y^3\, dy = z^2$$

$$\Rightarrow g(z) = \frac{\partial G(z)}{\partial z} = \begin{cases} 2z & 0 < z < 1 \\ 0 & \text{else} \end{cases}$$

Example 2:

$$f(x,y) = \begin{cases} 6(1-x-y) & x+y<1,\ x>0,\ y>0 \\ 0 & \text{else} \end{cases}$$

Find 1) $Pr(2X + 3Y < 1)$

2) $Pr(\frac{1}{2}X + 2Y < 1)$

$x = 2-4y$

1) $Pr(2X + 3Y < 1) = P(X < \frac{1}{2} - \frac{3}{2}y)$

$\Rightarrow x < \frac{1}{2}$, $y < \frac{1}{3}$

$$= \int_0^{1/3} \int_0^{\frac{1}{2} - \frac{3}{2}y} 6(1-x-y)\, dx\, dy$$

$\frac{1}{2}x + 2y = 1$
$2 - 4y + y = 1$
$2 - 3y = 1$
$3y = 1$
$y = \frac{1}{3}$

2) $Pr(\frac{1}{2}X + 2Y < 1) = Pr(X < 2-4y)$

$$= \int_0^{1/3} \int_0^{1-y} f(x,y)\, dx\, dy + \int_{1/3}^{1/2} \int_0^{2-4y} f(x,y)\, dx\, dy$$

2 terms to integral

Special distributions:

$\underbrace{B(n,p)}$ vs $\underbrace{NB(r,p)}$

$X \equiv$ # successes in n experiments

$Y \equiv$ # failures before exactly r results

$$f(y) = \underbrace{\binom{y+r-1}{r-1} p^{r-1}(1-p)^y}_{g_{bin(y+r-1,p)}(r-1)} \cdot p$$

$$= \binom{y+r-1}{r-1} p^r (1-p)^y$$

$M(t) = [1-p+pe^t]^n$ binomial

If $M(t) = (a+be^t)^n$, $a+b=1$, then we must have a binomial(n,b) random variable.

$M(t) = (\frac{2}{3} + \frac{1}{3}e^t)^5$

$E[X] = \frac{5}{3}$

$Var(X) = \frac{10}{9}$

Taylor Series expansion

$$f(u) = f(u_0) + f'(u_0)(u-u_0) + \frac{f''(u_0)}{2!}(u-u_0)^2 + \dots + \frac{f^{(n)}(u_0)}{n!}(u-u_0)^n \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(u_0)}{n!}(u-u_0)^n$$

$$\Rightarrow e^u = e^{u_0} + e^{u_0}(u-u_0) + \frac{e^{u_0}}{2!}(u-u_0)^2 + \frac{e^{u_0}}{3!}(u-u_0)^3 + \dots$$

If $u_0 = 0$, then $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$

$$e^m = \sum_{x=0}^{\infty} \frac{m^x}{x!} \Rightarrow 1 = \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!}$$

$M(t) = \exp[m(e^t - 1)]$

$E[X] = m$

$Var(X) = m$

ex:

Suppose MGF of X is $e^{4(e^t-1)}$

$m = 4$, $\mu = 4$, $\sigma = 2$

$\Rightarrow Pr(\mu - 2\sigma < X < \mu + 2\sigma) = 0.931$

$$Pr(0 < X < 8) = \sum_{x=1}^{7} \frac{e^{-4} 4^x}{x!} = 0.931$$

Suppose $X \sim$ poisson(100)

Determine a lower bound for

$$P_r(75 < X < 125) = P_r(|X - 100| \leq 2.5\mu) \geq 1 - \frac{1}{6.25}$$

$$= \frac{625 - 100}{625} = \frac{525}{625} = \frac{21}{25}$$

Summary of MGF

$B(n, p)$: $(1 - p + pe^t)^n$

Poisson: $\exp(m(e^t - 1))$

Gamma: $(1 - \beta t)^{-\alpha}$

Normal: $\exp(\mu t + \frac{\sigma^2 t^2}{2})$

Example. Suppose X is a random variable with $E[X^m] = (m+1)!\,2^m$

What is the MGF and dist of X

$$E[X^m] = (m+1)!\,2^m = \left.\frac{d^m M(t)}{dt^m}\right|_{t=0}$$

Gamma: $\frac{d^m M(0)}{dt^m} = (m + \alpha - 1) \cdots (\alpha+1)(\alpha)\,\beta^m$

Thus, $\alpha = \beta = 2$ by uniqueness of MGF.