Large sample theory "convergence"

$Y_1, Y_2, Y_3, \ldots$ a sequence of rvs,

Defn: Y_n converges in probability to Y if

$$\lim_{n\to\infty} \underbrace{\Pr[|Y_n - Y| \geq \varepsilon]}_{\text{this is a nonstochastic creature}} = 0 \quad \forall \varepsilon > 0$$

"high school limit"

alternatively, we write $\operatorname{plim}_{n\to\infty} Y_n = Y$

$(\Omega, \mathcal{C}, P)$

Defn: We say Y_n converges almost surely to Y if

$$P[\{\omega : \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\}] = 1$$

fix ω. Then $Y_n(\omega)$ is nonstochastic. $\lim_{n\to\infty} Y_n(\omega) = Y(\omega)$

says $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \ni \forall n \geq N, \; |Y_n(\omega) - Y(\omega)|$

Do this $\forall \omega$. If the size of $\{\omega\}$ that satisfy this is one in the P measure.

Then we say $Y_n \to Y$ a.s.

Theorem: If $Y_n \to Y$ a.s., then $\operatorname{plim} Y_n = Y$.

Suppose we have $\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix}$ a.s., Then

$X_n + Y_n \to X + Y$ in probability

Let $\Omega_1 = \{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega), \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\}$

$P(\Omega_1) = 1$ by assumption.

If $\omega \in \Omega_1$, then $\lim_{n\to\infty} (X_n(\omega) + Y_n(\omega)) = X(\omega) + Y(\omega)$

Let $\Omega_2 = \{\omega : \lim_{n\to\infty} (X_n(\omega) + Y_n(\omega)) = X(\omega) + Y(\omega)\}$

If $\omega \in \Omega_1$, then $\omega \in \Omega_2 \Rightarrow \Omega_1 \subset \Omega_2$. Thus, $P(\Omega_2) = 1$

Therefore $X_n + Y_n \to X + Y$ almost surely

$\Rightarrow \operatorname{plim}(X_n + Y_n) = X + Y$

Strong Law of Large Numbers

Suppose $Y_1, Y_2, \dots$ are iid random vectors, $E[|Y_i|] < +\infty$.
Consider $\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Then $\overline{Y}_n \rightarrow E[Y_i]$ a.s.

Weak Law of Large Numbers

Under the same conditions, $\text{plim } \overline{Y}_n = E[Y_i]$

Monte Carlo simulation:

$$Z = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \qquad Y = u(Z), \text{ some function}$$

random vector

What is the cdf of Y? What about the moments? It is too complicated sometimes.

Computer can generate iid rvs Z and plug them into $u(Z)$.

$U \sim$ uniform $(0,1)$ generated by computer.

$$\text{pdf} \quad f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

Take $y \in \mathbb{R}$. $\Pr(u(Z) \le y) = \Pr[Y \le y]$?
If we can find this $\forall y \in \mathbb{R}$, we know everything. While we cannot exactly characterize this, the computer and SLLN can help

Note that $\Pr[u(Z) \le y] = E[\mathbb{1}(u(Z) \le y)]$

Computer can generate $Z_1, Z_2, \dots$ iid w/ same distribution as Z.
Since $Z_1, Z_2, \dots$ iid, then $\mathbb{1}(u(Z_1) \le y), \mathbb{1}(u(Z_2) \le y), \dots$ are iid.

From WLLN, $\underset{N\to\infty}{\text{plim}} \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}(u(Z_i) \le y) = E[\mathbb{1}(u(\underset{\substack{\| \\ Z\ \forall i \text{ by iid}}}{Z_i}) \le y)] = \Pr[u(Z) \le y]$

In effect, you have nailed down the pdf of $Y = u(Z)$.

Suppose you want to find $E[u(Z_i)]$?

Just do: $\underset{N\to\infty}{\text{plim}} \frac{1}{N}\sum_{i=1}^{N} u(Z_i) = E[u(Z_i)]$

From a uniform rv U, you can generate a rv with target cdf: F which is strictly increasing.

Define $Y \equiv F^{-1}(U) \Rightarrow$ cdf of Y is F.

Pf: $\Pr(Y \le y) = \Pr(F^{-1}(U) \le y)$

$= \Pr(F(F^{-1}(U)) \le F(y))$ which holds since F and F^{-1} are strictly inc.

$= \Pr(U \le \underset{\in(0,1)}{F(y)})$

Recall that $f(u) = \begin{cases} 1 & u \in (0,1) \\ 0 & \text{else} \end{cases}$ pdf

Thus, $\Pr(Y \le y) = \int_0^{F(y)} 1\, du = F(y)$ $\square$