1) Contraction mapping theorem
2) Measurable functions and σ-algebras
3) Midterm 2003
4) Solow residual
5) Question 3 on Winter 2005 midterm

an operator is a mapping $T: X \to Y$

eg. $\underset{m\times n}{A}\ \underset{n\times 1}{v} = \underset{m\times 1}{w} \Rightarrow A: \mathbb{R}^n \to \mathbb{R}^m$

Defn: a contraction T of modulus β is an operator s.t. $\rho(Tx, Ty) \le \beta\, \rho(x,y)\quad \forall x,y \in X,\ \beta \in (0,1)$.

$T: \mathbb{R} \to \mathbb{R}$, $x \mapsto \gamma x + b$ where $|\gamma| < 1$. Let $\rho(x,y) = |x-y|$

x_0 given
$x_1 = Tx_0$
$\vdots$
$x_t = T^t x_0$

Show that this is a contraction

$$\rho(Tx, Ty) = |\gamma x + b - (\gamma y + b)| = |\gamma x - \gamma y| = |\gamma|\,|x-y|$$
$$\le |\gamma|\, \rho(x,y) \quad \text{where } |\gamma| < 1.$$

Thus, T is a contraction of modulus $|\gamma|$

Thm: If T is a contraction, then $\exists!\ \hat{x}$ satisfying $\hat{x} = T\hat{x}$

Example of something is not a contraction

$S: \mathbb{R} \to \mathbb{R}$, $x \mapsto ax^2 + (1+b)x + c$

Let $\hat{x} = S\hat{x} = a\hat{x}^2 + (1+b)\hat{x} + c \Rightarrow a\hat{x}^2 + b\hat{x} + c = 0$

$\Rightarrow \hat{x} \in \left\{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right\}$ if $\left|\left\{\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right\}\right| = 2$, then S has multiple fixed points $\Rightarrow S$ is not a contraction.

Defn: $\mathcal{X}$ is a σ-algebra on $\mathbb{X}$ if

i) $\emptyset \in \mathcal{X}$

ii) $A \in \mathcal{X} \Rightarrow A^c \in \mathcal{X}$

iii) $A_i \in \mathcal{X},\ i=1,2,\ldots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{X}$.

Defn: $(\mathbb{X}, \underset{\sigma\text{-algebra}}{\mathcal{X}})$ is a measurable space.

Defn: a function $f: \mathbb{X} \to \mathbb{R}$ is measurable wrt the measurable space $(\mathbb{X}, \mathcal{X})$ if $\forall a \in \mathbb{R},\ \{x : f(x) \le a\} \in \mathcal{X}$.

Let $\mathbb{X} = \{1,2,3,4\}$

$\mathcal{X} = \{\emptyset, \{1,2\}, \{3,4\}, \{1,2,3,4\}\}$. $\mathcal{X}$ is a σ-algebra on $\mathbb{X}$.

$(\mathbb{X}, \mathcal{X})$ is thus a measurable space

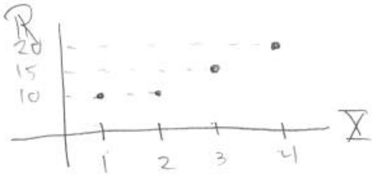
Let $f: \mathbb{X} \to \mathbb{R}$ be such that

$1 \overset{f}{\mapsto} 10$

$2 \overset{f}{\mapsto} 10$

$3 \overset{f}{\mapsto} 15$

$4 \overset{f}{\mapsto} 20$



$a \ge 20 \Rightarrow \{x : f(x) \le a\} = \{1,2,3,4\}$

$15 \le a < 20 \Rightarrow \{x : f(x) \le a\} = \{1,2,3\} \notin \mathcal{X}$

$10 \le a < 15 \Rightarrow \{x : f(x) \le a\} = \{1,2\}$

$a < 10 \Rightarrow \{x : f(x) \le a\} = \{\emptyset\}$

Thus $\exists\, a \in \mathbb{R} \ni \{x : f(x) \le a\} \notin \mathcal{X} \Rightarrow f$ is not measurable wrt. $\mathcal{X}$.

Let $f': \mathbb{X} \to \mathbb{R}$

$1 \mapsto 10$ $\quad a \ge 20 \Rightarrow \{1,2,3,4\}$

$2 \mapsto 10$ $\quad 10 \le a < 20 \Rightarrow \{1,2\}$

$3 \mapsto 20$ $\quad a < 10 \Rightarrow \{\emptyset\}$

$4 \mapsto 20$

Thus $\forall a \in \mathbb{R},\ \{x : f'(x) \le a\} \in \mathcal{X} \Rightarrow f'$ is measurable wrt $\mathcal{X}$.

2a) a vectorspace is a set V and operations $(\cdot, +)$ satisfying
i) $x, y \in V \Rightarrow x+y \in V$
ii) $x \in V, \alpha \in \mathbb{R} \Rightarrow \alpha x \in V$
iii) $\exists \theta \in V \ni x + \theta = x \quad \forall x \in V$; also iii) $0x = \theta$
iv) $1x = x \quad \forall x \in V$ where $1 \in \mathbb{R}$.

b) Let $S = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, $a, b \geq 0$, $a, b \in \mathbb{R}$
No. $-1 \cdot x \notin S \quad \forall x \in S \Rightarrow$

c) Let $C[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ continuous}\}$
i) Let $g, f \in C[a,b], \alpha, \beta \in \mathbb{R} \Rightarrow h(x) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) \in C[a,b]$
ii) $\theta(x) = 0 \quad \forall x$, $\theta \in C[a,b]$. Then $\underbrace{\theta(x)}_{0} + f(x) = f(x) \quad \forall x \in [a,b]$
iii) $1 \cdot f = f \quad \forall f \in C[a,b]$
Thus, $C[a,b]$ is a vector space

3a) a metric space is a set (X, ρ) satisfying:
i) $\rho(x,y) > 0 \quad x \neq y$, $\rho(x,x) = 0$
ii) $\rho(x,y) = \rho(y,x)$
iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

b) Let $S = [a,b]$, $\rho(x,y) = |x-y|$. By last section, (S, ρ) is a metric space.

c) $(C[a,b], \rho)$ where $\rho(x,y) = \max_{a \leq t \leq b} |x(t) - y(t)|$. Is this a metric space?
i) Clearly, $\rho(x,y) = \rho(y,x)$
ii) Clearly, $\rho(x,x) = 0$ and $\rho(x,y) > 0$
iii)
$$\begin{aligned}
\rho(x,y) + \rho(y,z) &= \max_{a \leq t \leq b} |x(t) - y(t)| + \max_{a \leq t \leq b} |y(t) - z(t)| \\
&\geq \max_{a \leq t \leq b} \{|x(t) - y(t)| + |y(t) - z(t)|\} \\
&\geq \max_{a \leq t \leq b} \{|(x(t) - y(t)) + (y(t) - z(t))|\} \\
&= \max_{a \leq t \leq b} |x(t) - z(t)| = \rho(x,z)
\end{aligned}$$

Therefore, $(C[a,b], \rho)$ is a metric space.

d) $S = \{x : 0 \le x \le 1\}$

$\Sigma = \{[0,1], [0, \frac{1}{2}), [\frac{1}{2}, 1]\}$

Is (S, Σ) measurable? $\Leftrightarrow$ is Σ a σ-algebra? $\emptyset \notin \Sigma$

Solow Residual:

$$Y_t = A_t K_t^{\alpha} \left((1+g)^t L_t \right)^{1-\alpha}$$

$$\ln Y_t = \ln A_t + \alpha \ln K_t + (1-\alpha) \ln L_t + t(1-\alpha)\ln(1+g)$$

Here, $\ln A_t + t(1-\alpha)\ln(1+g) \equiv SR_t$ is the Solow residual

$$\equiv \ln Y_t - \alpha \ln K_t - (1-\alpha)\ln L_t$$

$$\left[w_t = \frac{\partial Y_t}{\partial L_t} = (1-\alpha)\frac{Y_t}{L_t}, \quad r_t = \frac{\partial Y_t}{\partial K_t} = \alpha \frac{Y_t}{K_t} \right.$$

We have time series for w_t, r_t, Y_t, L_t, and K_t. From this, we can derive α, $(1-\alpha)$

Question 3 from Winter 2005

$$\underset{n\times n}{A}\ \underset{n\times 1}{y_t} = \underset{n\times n}{B}\ \underset{n\times 1}{y_{t-1}} + \underset{n\times 1}{C} + \underset{n\times k}{\Psi}\ \underset{k\times 1}{u_t} + \underset{n\times m}{\Pi}\ \underset{m\times 1}{\varepsilon_t}$$

a) we have $A = QSZ$, $B = QTZ$ where $QQ' = I = ZZ'$ and S, T are upper triangular

b) assume $\text{rank}(A) < n \Rightarrow A$ not invertible
$\text{rank}(B) < n \Rightarrow B$ not invertible

$$QSZy_t = QTZy_{t-1} + C + \Psi u_t + \Pi \varepsilon_t$$

$$S\underset{\equiv}{Z}y_t = TZy_{t-1} + Q'C + Q'\Psi u_t + Q'\Pi \varepsilon_t$$

Define $Zy_t = p_t$

$$S p_t = T p_{t-1} + Q'C + Q'\Psi u_t + Q'\Pi \varepsilon_t$$

Suppose we have ℓ stable roots and $n-\ell$ unstable roots

Thus, we have $n-\ell$ "boundedness" conditions and ℓ initial conditions.

$$S \begin{bmatrix} p_t^1 \\ \vdots \\ p_t^{\ell} \\ p_t^{\ell+1} \\ \vdots \\ p_t^n \end{bmatrix} = T \begin{bmatrix} p_{t-1}^1 \\ \vdots \\ p_{t-1}^{\ell} \\ \hline p_{t-1}^{\ell+1} \\ \vdots \\ p_{t-1}^n \end{bmatrix} + Q'C + Q'\Psi u_t + Q'\Pi \varepsilon_t$$

Also, $m = n - \ell$ is necessary to pin down the endogenous shocks.