Annie, Chris

Markov Chains and Markov Processes

Important concepts:

① a measure space is a triple $(X, \mathcal{X}, \lambda)$ where $\mathcal{X}$ is a σ-algebra on X and μ is a σ-additive extended real-valued fcn on $\mathcal{X}$.

② a measure is a fcn $\mu: \mathcal{X} \to \overline{\mathbb{R}}$ satisfying

i) $\mu(\emptyset) = 0$

ii) $\mu(A) \geq 0 \quad \forall A \in \mathcal{X}$

iii) If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{X} \ni A_i \cap A_j = \emptyset \ \forall i \neq j$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

If $\mu(X) = 1$, then we say that μ is a probability measure.

If any property holds $\forall x \in X$ and $\forall A \in \mathcal{X}$ except on sets of measure zero, then we say that holds almost everywhere.

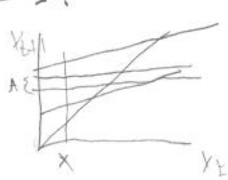
Defn: a measurable function is a function $f: X \to \overline{\mathbb{R}}$ such that $\{x \mid f(x) \leq a\} \in \mathcal{X} \quad \forall a \in \mathbb{R}$.

Defn: a random variable is a measurable function on a probability space $(X, \mathcal{X}, \mu)$

Defn: a stochastic process is a family of random variables $\{x_t\}_{t=1}^{\infty}$.

Defn: Let $p(s, A)$ be the probability that $x_{t+1} \in A$ given that $x_t = s$.

Eg.



Defn: a Markov process is a stochastic process for which $p(x_t, A_{t+1})$ does not depend on x_s, $s \leq t-1$.

Defn: a Markov chain is a Markov process defined on a finite space $\mathbb{X}$.

Markov chains:

a Markov matrix P is a matrix with the property that

i) $p_{ij} \geq 0 \quad \forall i, j$

ii) $\sum_{j=1}^{n} p_{ij} = 1$

Let the state space be $S = \{s_1, \ldots, s_n\}$. Denote

$$p_{ij} = Pr(x_{t+1} = s_j \mid x_{t+1} = s_i)$$

Example: $s_1 = 1$, $s_2 = 2$, $s_3 = 3$

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Let $\pi_1 = \begin{bmatrix} \pi_{11} \\ \pi_{12} \\ \pi_{13} \end{bmatrix}$ where $\pi_{1k} = Pr(x_1 = s_k)$

In general, $\pi_t' = \pi_{t-1}' P$

For our example, suppose $\pi_1 = \begin{bmatrix} \pi_{11} \\ \pi_{12} \\ \pi_{13} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$. Then,

$$\pi_2' = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \pi_{21} & \pi_{22} & \pi_{23} \end{bmatrix}$$

i.e. $\pi_{21} = Pr(x_1 = 1) Pr(x_2 = 1 \mid x_1 = 1) + Pr(x_1 = 2) Pr(x_2 = 1 \mid x_1 = 2) + Pr(x_1 = 3) Pr(x_2 = 1 \mid x_1 = 3)$

$\quad = \pi_{11} p_{11} + \pi_{12} p_{21} + \pi_{13} p_{31}$

In general, $\pi_{t+1,k} = Pr(x_{t+1} = k) = \sum_{j=1}^{n} Pr(x_{t+1} = k \mid x_t = j) Pr(x_t = j)$

$$= \sum_{j=1}^{n} p_{jk} \pi_{tj}$$

In general, we have: $\pi_t' = \pi_0' P^t$

Defn: an <u>invariant probability distribution</u> is a distribution π^* satisfying $(\pi^*)' = (\pi^*)' P$

Defn: an <u>ergodic set</u> E is a set E with $p(s, E) = 1$ $\forall s \in E$ and there is no proper subset of E that has this property.

Theorem: If $p_{ij} > 0$ $\forall i j$, then

① $\exists$ a unique invariant distribution π^* with no cyclically moving subsets

② $\pi_0' P^t \xrightarrow[t\to\infty]{} (\pi^*)'$ $\forall \pi_0$

Theorem: P has at least one unit eigenvalue.

Pf: Define $\underset{1\times n}{i} = (1 \ldots 1)$. Notice that $i P' = i$

$$\underset{1\times n}{0} = i \cdot 0 = \underset{1\times n}{i}\,(\underset{n\times n}{P'} - \lambda_j \underset{n\times n}{I})\, x_j = (iP' - \lambda_j i I)\, x_j$$
$$= (i - \lambda_j i)\, x_j$$
$$= (1 - \lambda_j)\, \underbrace{i x_j}_{>0}$$

Therefore $\lambda_j = 1$ for some j. (Recall that the eits of P are the same as the eits of P')

Markov transition matrices and conditional expectations:

Consider any function f.

Eg. $f(x) = x^2$ with $S = \{1, 2, 3\}$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1^2 \\ 2^2 \\ 3^2 \end{bmatrix} = \begin{bmatrix} \frac{13}{2} \\ \frac{5}{2} \\ \frac{10}{2} \end{bmatrix}$$

More generally, see top of next page:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} p_{11} f(1) + p_{12} f(2) + p_{13} f(3) \\ p_{21} f(1) + p_{22} f(2) + p_{23} f(3) \\ p_{31} f(1) + p_{32} f(2) + p_{33} f(3) \end{bmatrix} = \begin{bmatrix} E[f(x_{t+1}) \mid x_t = 1] \\ E[f(x_{t+1}) \mid x_t = 2] \\ E[f(x_{t+1}) \mid x_t = 3] \end{bmatrix}$$

Markov processes

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space. Let Λ be a family of probability measures.

$Q(x, A)$ is a Markov transition function if (1) $\forall x \in \mathbb{X}$, $Q(x, A)$ is a probability measure for fixed $A \in \mathcal{X}$ and (2) $\forall A \in \mathcal{X}$, $Q(x, A)$ is a measurable function for fixed $x \in \mathbb{X}$

$Q(x, A)$ is a measurable function

where $Q(x, A) = Pr(x_{t+1} \in A \mid x_t = x)$ $\left\{ Q: S \times \mathcal{X} \to \mathbb{R} \cap [0,1],\ (x, A) \mapsto Q(x, A) \right\}$

If $\lambda_t(A)$ is a probability measure that represents $P(x \in A)$ at date t. Then $\lambda_{t+1}(A) = \underbrace{(T^* \lambda_t)}_{\text{new probability measure}}(A)$

where $T^* \lambda_t(A) = \int_{x \in \mathbb{X}} \underbrace{Q(x, A)}_{\substack{\text{Probability of} \\ \text{being at } x \text{ and} \\ \text{going into } A}} \underbrace{d\lambda_t}_{\substack{\text{probability} \\ \text{of being} \\ \text{at } x \text{ at } t}}$

The operator T finds conditional expectations of measurable functions

$$E[f(x_{t+1} \mid x_t)] = \int Q(x_t, dx_{t+1}) f(x_t)$$

$$\equiv (Tf)(x)$$